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# Stable periodic solution of a discrete periodic Lotka–Volterra competition system

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**Abstract**

This paper discusses a discrete Lotka–Volterra competition system. We first obtain the persistence of the system. Assuming that the coefficients in the system are periodic, we obtain the existence of a periodic solution. Moreover, under some additional conditions, this periodic solution is globally stable. Our results not only reduce to those for the scalar equation when there is no coupling but also improve and complement some in the literature.


*Keywords:* Lotka–Volterra competition system; Persistence; Periodic solution; Global stability

## 1. Introduction

Consider the following system of difference equations,

$$\begin{cases} x(n+1) = x(n)\exp\Bigl[r_1(n)\Bigl(1 - \frac{x(n)}{K_1(n)} - \mu_2(n)y(n)\Bigr)\Bigr], \\ y(n+1) = y(n)\exp\Bigl[r_2(n)\Bigl(1 - \mu_1(n)x(n) - \frac{y(n)}{K_2(n)}\Bigr)\Bigr], \end{cases} \tag{1}$$

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where, for $i = 1$ and 2, $\{r_i(n)\}$, $\{K_i(n)\}$ and $\{\mu_i(n)\}$ are bounded nonnegative sequences such that

$$0 < K_{i*} \leqslant K_i^*, \qquad 0 < r_{i*} \leqslant r_i^*, \qquad 0 \leqslant \mu_i^*. \tag{2}$$

Here, for any bounded sequence $\{a(n)\}$, $a^* = \sup_{n\in\mathbb{N}} a(n)$ and $a_* = \inf_{n\in\mathbb{N}} a(n)$. System (1) is a discrete two-species competition model of Lotka–Volterra type (see, for example, May [12]). From the point of view of biology, in the sequel, we assume that $x(0) > 0$ and $y(0) > 0$. Then system (1) has a positive solution $(x(n), y(n))_{n=0}^{\infty}$ passing through $(x(0), y(0))$.

Since May [12], a lot of work has been done for system (1) and some generalized systems of Lotka–Volterra type. It has been found that such systems can demonstrate quite rich and complicated dynamics such as limit cycle, various bifurcation and even chaotic oscillation. See, to name a few, [4,8,11,13,14] and the references therein. We should mention that all the systems considered in the aforementioned references are autonomous. However, in many situations, $\{r_i(n)\}$, $\{K_i(n)\}$ and $\{\mu_i(n)\}$ can be assumed to be non-constant sequences. For example, assuming that they are periodic accounts for the seasonal fluctuations. In this paper, we will consider the nonautonomous case.

Note that system (1) is a result of coupling of two scalar equations of the form

$$x(n+1) = x(n)\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right]. \tag{3}$$

Eq. (3) has been studied recently by Zhou and Zou [18]. Sufficient conditions on persistence and sufficient conditions on the existence and stability of a periodic solution are obtained. It is natural to expect similar results for (1) which reduce to those in [18] when coupling disappears (i.e., $\mu_1(n) \equiv \mu_2(n) \equiv 0$). This is the goal of this paper.

In theoretical ecology, it is important whether or not all species in a multispecies community can be persistent. Though much has been done for persistence of models governed by differential equations in the literature (see [1–3,6,9,10,15,17] and the references therein), there are only several papers (see, for example, [5,7,11,14]) on the persistence of discrete models of Lotka–Volterra type. However, the systems considered in the aforementioned references are autonomous. As mentioned earlier, it is more realistic to model the population growth by nonautonomous difference equations. One objective of this paper is to give sufficient conditions for the persistence of system (1). System (1) is said to be persistent if there is a compact subset $E$ in the interior of $\mathbb{R}_+^2$ such that each solution will eventually enter and remain in $E$.

A hallmark of observed population densities in the field is their oscillatory behavior. A main purpose of modeling population interactions is to understand what cause such fluctuations. One approach is to assume that the per capita growth function is time dependent and periodic in time. Another objective of this paper is to establish the existence of a periodic solution for system (1) under the assumption that all $\{r_i(n)\}$, $\{K_i(n)\}$ and $\{\mu_i(n)\}$ are periodic and establish the stability of the periodic solution.

When there is no coupling, the results obtained in this paper almost reduce to those obtained in [18] for (3). Moreover, these results also improve those obtained in [11] and complement those obtained in [16]. The organization of the paper is as follows. In the next section, we establish the persistence of system (1). Basing on the persistence result, under the assumptions of periodicity of $\{K_i(n)\}$, $\{r_i(n)\}$ and $\{\mu_i(n)\}$, we show the existence and stability of a periodic solution to system (1) in Section 3. We conclude this paper by some remarks and discussions.

## 2. Persistence

In this section, we establish a persistence result for system (1).

**Proposition 1.** *For every solution* $(x(n), y(n))$ *of* (1) *we have*

$$\limsup_{n\to\infty} x(n) \leqslant x^* \quad \textit{and} \quad \limsup_{n\to\infty} y(n) \leqslant y^*, \tag{4}$$

*where* $x^* = \frac{K_1^*}{r_1^*}\exp(r_1^* - 1)$ *and* $y^* = \frac{K_2^*}{r_2^*}\exp(r_2^* - 1)$.

**Proof.** We only need to show that

$$\limsup_{n\to\infty} x(n) \leqslant x^* \tag{5}$$

since similar result can be shown for $y(n)$ and then (4) follows obviously.

To prove (5), we first assume that there exists an $l_0 \in \mathbb{N}$ such that $x(l_0+1) \geqslant x(l_0)$. Then

$$1 - \frac{x(l_0)}{K_1(l_0)} - \mu_2(l_0)y(l_0) \geqslant 0.$$

Hence, $x(l_0) \leqslant K_1(l_0) \leqslant K_1^*$, which is less than $x^*$ since $\frac{\exp(r_1^*-1)}{r_1^*} \geqslant 1$. It follows that

$$\begin{aligned} x(l_0+1) &= x(l_0)\exp\left[r_1(l_0)\left(1 - \frac{x(l_0)}{K_1(l_0)} - \mu_2(l_0)y(l_0)\right)\right] \\ &\leqslant x(l_0)\exp\left[r_1^*\left(1 - \frac{x(l_0)}{K_1(l_0)}\right)\right] = K_1(l_0)\frac{x(l_0)}{K_1(l_0)}\exp\left[r_1^*\left(1 - \frac{x(l_0)}{K_1(l_0)}\right)\right] \\ &\leqslant \frac{K_1^*}{r_1^*}\exp\bigl(r_1^* - 1\bigr) = x^*, \end{aligned}$$

here we used $\max_{x\in\mathbb{R}} x\exp(r(1-x)) = \frac{\exp(r-1)}{r}$ for $r > 0$. We claim that

$$x(n) \leqslant x^* \quad \text{for } n \geqslant l_0.$$

By way of contradiction, assume that there exists a $p_0 > l_0$ such that $x(p_0) > x^*$. Then $p_0 \geqslant l_0 + 2$. Let $\tilde{p}_0 \geqslant l_0 + 2$ be the smallest integer such that $x(\tilde{p}_0) > x^*$. Then $x(\tilde{p}_0 - 1) < x(\tilde{p}_0)$. The above argument produces that $x(\tilde{p}_0) \leqslant x^*$, a contradiction. This proves the claim. Now, we assume that $x(n+1) < x(n)$ for all $n \in \mathbb{N}$. In particular, $\lim_{n\to\infty} x(n)$ exists, denoted by $\bar{x}$. We claim that $\bar{x} \leqslant K_1^*$. By way of contradiction, assume that $\bar{x} > K_1^*$. Taking limit in the first equation in system (1) gives

$$\lim_{n\to\infty}\left(1 - \frac{x(n)}{K_1(n)} - \mu_2(n)y(n)\right) = 0,$$

which is a contradiction since

$$1-\frac{x(n)}{K_1(n)}-\mu_2(n)y(n)\leqslant 1-\frac{x(n)}{K_1(n)}\leqslant 1-\frac{\bar{x}}{K_1^*}<0 \quad \text{for } n\in\mathbb{N}.$$

This proves the claim. Note that $K_1^*\leqslant x^*$. It follows that (5) holds. This completes the proof. □

Proposition 1 implies that solutions of (1) are bounded eventually. Moreover, from the proof, we only need the upper boundedness of $\{K_i(n)\}$ and $\{r_i(n)\}$ and need the lower bound of $\{r_i(n)\}$ to be larger than 0 to guarantee the boundedness of solutions. For system (1) to be persistent, we also need the lower bound of $\{K_i(n)\}$ to be larger than 0 and the two species are not in high competition as the proof of the following proposition shows.

**Proposition 2.** *Assume that* $1-\mu_1^*x^*>0$ *and* $1-\mu_2^*y^*>0$ *where* $x^*$ *and* $y^*$ *are the same as in Proposition* 1. *Then*

$$\liminf_{n\to\infty}x(n)\geqslant x_* \quad \textit{and} \quad \liminf_{n\to\infty}y(n)\geqslant y_*, \tag{6}$$

*where* $x_*=K_{1*}(1-\mu_2^*y^*)\exp[r_1^*(1-\mu_2^*y^*-\frac{x^*}{K_{1*}})]$ *and* $y_*=K_{2*}(1-\mu_1^*x^*)\exp[r_2^*(1-\mu_1^*x^*-\frac{y^*}{K_{2*}})]$.

**Proof.** Again we only need to show that

$$\liminf_{n\to\infty}x(n)\geqslant x_*. \tag{7}$$

For any $\varepsilon>0$ which satisfies

$$1-\mu_1^*(x^*+\varepsilon)>0 \quad \text{and} \quad 1-\mu_2^*(y^*+\varepsilon)>0,$$

according to Proposition 1, there exists $n^*\in\mathbb{N}$ such that

$$x(n)\leqslant x^*+\varepsilon \quad \text{and} \quad y(n)\leqslant y^*+\varepsilon \quad \text{for } n\geqslant n^*.$$

First, we assume that there exists an $l_0\geqslant n^*$ such that $x(l_0+1)\leqslant x(l_0)$. Note that, for $n\geqslant l_0$,

$$\begin{aligned}x(n+1)&=x(n)\exp\left[r_1(n)\left(1-\frac{x(n)}{K_1(n)}-\mu_2(n)y(n)\right)\right]\\&\geqslant x(n)\exp\left[r_1(n)\left(1-\mu_2^*(y^*+\varepsilon)-\frac{x(n)}{K_1(n)}\right)\right].\end{aligned}$$

In particular, with $n=l_0$, we get

$$1-\mu_2^*(y^*+\varepsilon)-\frac{x(l_0)}{K_1(l_0)}\leqslant 0,$$

which implies that $x(l_0)\geqslant K_{1*}(1-\mu_2^*(y^*+\varepsilon))$. Then

$$\begin{aligned}
x(l_0+1) &\geqslant K_{1*}\big(1-\mu_2^*(y^*+\varepsilon)\big)\exp\left[r_1(l_0)\left(1-\mu_2^*(y^*+\varepsilon)-\frac{x(l_0)}{K_1(l_0)}\right)\right]\\
&\geqslant K_{1*}\big(1-\mu_2^*(y^*+\varepsilon)\big)\exp\left[r_1^*\left(1-\mu_2^*(y^*+\varepsilon)-\frac{x(l_0)}{K_1(l_0)}\right)\right]\\
&\geqslant K_{1*}\big(1-\mu_2^*(y^*+\varepsilon)\big)\exp\left[r_1^*\left(1-\mu_2^*(y^*+\varepsilon)-\frac{x^*+\varepsilon}{K_{1*}}\right)\right].
\end{aligned}$$

Let

$$x_\varepsilon = K_{1*}\big(1-\mu_2^*(y^*+\varepsilon)\big)\exp\left[r_1^*\left(1-\mu_2^*(y^*+\varepsilon)-\frac{x^*+\varepsilon}{K_{1*}}\right)\right].$$

We claim that

$$x(n) \geqslant x_\varepsilon \quad \text{for } n \geqslant l_0. \tag{8}$$

By way of contradiction, assume that there exists a $p_0 \geqslant l_0$ such that $x(p_0) < x_\varepsilon$. Then $p_0 \geqslant l_0 + 2$. Let $\tilde{p}_0 \geqslant l_0 + 2$ be the smallest integer such that $x(\tilde{p}_0) < x_\varepsilon$. Then $x(\tilde{p}_0 - 1) > x(\tilde{p}_0)$. The above argument produces that $x(\tilde{p}_0) \geqslant x_\varepsilon$, a contradiction. This proves the claim. Now, we assume that $x(n+1) > x(n)$ for all large $n$. Then $\lim_{n\to\infty} x(n)$ exists, denoted by $\underline{x}$. We claim that $\underline{x} \geqslant K_{1*}(1-\mu_2^*(y^*+\varepsilon))$. By way of contradiction, assume that $\underline{x} < K_{1*}(1-\mu_2^*(y^*+\varepsilon))$. Taking limit in the first equation in system (1) gives

$$\lim_{n\to\infty}\left(1-\frac{x(n)}{K_1(n)}-\mu_2(n)y(n)\right)=0,$$

which is a contradiction since

$$\liminf_{n\to\infty}\left(1-\frac{x(n)}{K_1(n)}-\mu_2(n)y(n)\right)\geqslant 1-\mu_2^*(y^*+\varepsilon)-\frac{\underline{x}}{K_{1*}}>0.$$

This proves the claim. Note that $x^* \geqslant K_1^* \geqslant K_{1*}$ implies $K_{1*}(1-\mu_2^*(y^*+\varepsilon)) \geqslant x_\varepsilon$ and $\lim_{\varepsilon\to 0} x_\varepsilon = x_*$. We can easily see that (7) holds. This completes the proof. □

Now, the main result of this section follows easily.

**Theorem 1.** *Assume* (2), $1-\mu_1^*x^* > 0$ *and* $1-\mu_2^*y^* > 0$ *hold. Then system* (1) *is persistent.*

It should be noticed that, from the proofs of Proposition 1 and Proposition 2, we know that under the condition of Theorem 1 the set $[x_*, x^*] \times [y_*, y^*]$ is an invariant set of system (1).

## 3. Existence and stability of a periodic solution

In this section, we consider system (1) with $\{r_i(n)\}$, $\{K_i(n)\}$ and $\{\mu_i(n)\}$ being periodic with a common period. More precisely, we assume that there exists a positive integer $\omega$ such that, for $n \in \mathbb{N}$ and for $i = 1$ and 2,

$$\begin{aligned}
&0 < r_i(n+\omega) = r_i(n), \\
&0 < K_i(n+\omega) = K_i(n), \\
&0 \leqslant \mu_i(n+\omega) = \mu_i(n). \qquad (9)
\end{aligned}$$

It follows from (9) that (2) holds. Let $x^*$ and $y^*$ be the same as in Proposition 1. Our first result concerns the existence of a periodic solution.

**Theorem 2.** *Assume that* (9) *holds. If* $1 - \mu_1^* x^* > 0$ *and* $1 - \mu_2^* y^* > 0$, *then system* (1) *has an $\omega$-periodic solution, denoted by* $(\tilde{x}(n), \tilde{y}(n))$.

**Proof.** Let $x_*$ and $y_*$ be the same as in Proposition 2. As noted at the end of the last section that $[x_*, x^*] \times [y_*, y^*]$ is an invariant set of (1). Thus, we can define a mapping $F$ on $[x_*, x^*] \times [y_*, y^*]$ by

$$F\bigl(x(0), y(0)\bigr) = \bigl(x(\omega), y(\omega)\bigr) \quad \text{for } \bigl(x(0), y(0)\bigr) \in [x_*, x^*] \times [y_*, y^*].$$

Obviously, $F$ depends continuously on $(x(0), y(0))$. Thus, $F$ is continuous and maps the compact set $[x_*, x^*] \times [y_*, y^*]$ into itself. Therefore, $F$ has a fixed point $(\tilde{x}, \tilde{y})$. It is easy to see that the solution, $(\tilde{x}(n), \tilde{y}(n))$ passing through $(\tilde{x}, \tilde{y})$ is an $\omega$-periodic solution of system (1). This completes the proof. □

Now, under some additional conditions, we study the global stability of the periodic solution obtained in Theorem 2.

**Theorem 3.** *Assume that* (9) *holds,* $1 - \mu_1^* x^* > 0$, $1 - \mu_2^* y^* > 0$ *and*

$$\begin{aligned}
\lambda_1 &= \max\left\{\left|1 - \frac{r_1^*}{K_{1*}} x^*\right|, \left|1 - \frac{r_{1*}}{K_1^*} x_*\right|\right\} + \mu_2^* r_1^* y^* < 1, \\
\lambda_2 &= \max\left\{\left|1 - \frac{r_2^*}{K_{2*}} y^*\right|, \left|1 - \frac{r_{2*}}{K_2^*} y_*\right|\right\} + \mu_1^* r_2^* x^* < 1. \qquad (10)
\end{aligned}$$

*Then for every solution* $(x(n), y(n))$ *of* (1), *we have*

$$\lim_{n\to\infty} \bigl(x(n) - \tilde{x}(n)\bigr) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \bigl(y(n) - \tilde{y}(n)\bigr) = 0, \qquad (11)$$

*where* $(\tilde{x}(n), \tilde{y}(n))$ *is the $\omega$-periodic solution obtained in Theorem* 2.

**Proof.** Let

$$x(n) = \tilde{x}(n)\exp\bigl(u(n)\bigr) \quad \text{and} \quad y(n) = \tilde{y}(n)\exp\bigl(v(n)\bigr).$$

Then (1) is equivalent to

$$\begin{cases}
u(n+1) = u(n) - \frac{r_1(n)\tilde{x}(n)}{K_1(n)}(\exp(u(n)) - 1) - r_1(n)\mu_2(n)\tilde{y}(n)(\exp(v(n)) - 1), \\
v(n+1) = v(n) - \frac{r_2(n)\tilde{y}(n)}{K_2(n)}(\exp(v(n)) - 1) - r_2(n)\mu_1(n)\tilde{x}(n)(\exp(u(n)) - 1).
\end{cases}$$

Therefore,

$$\begin{cases} u(n+1) = \left(1 - \frac{r_1(n)}{K_1(n)}\tilde{x}(n)\exp(\theta_1(n)u(n))\right)u(n) \\ \qquad\qquad - r_1(n)\mu_2(n)\tilde{y}(n)\exp(\theta_2(n)v(n))v(n), \\ v(n+1) = \left(1 - \frac{r_2(n)}{K_2(n)}\tilde{y}(n)\exp(\theta_2(n)v(n))\right)v(n) \\ \qquad\qquad - r_2(n)\mu_1(n)\tilde{x}(n)\exp(\theta_1(n)u(n))u(n), \end{cases} \tag{12}$$

where $\theta_1(n), \theta_2(n) \in [0,1]$. To complete the proof, it suffices to show that

$$\lim_{n\to\infty} u(n) = \lim_{n\to\infty} v(n) = 0. \tag{13}$$

In view of (10), we can choose $\varepsilon > 0$ such that

$$\lambda_1^\varepsilon = \max\left\{\left|1 - \frac{r_1^*}{K_{1*}}(x^* + \varepsilon)\right|, \left|1 - \frac{r_{1*}}{K_1^*}(x_* - \varepsilon)\right|\right\} + \mu_2^* r_1^*(y^* + \varepsilon) < 1,$$
$$\lambda_2^\varepsilon = \max\left\{\left|1 - \frac{r_2^*}{K_{2*}}(y^* + \varepsilon)\right|, \left|1 - \frac{r_{2*}}{K_2^*}(y_* - \varepsilon)\right|\right\} + \mu_1^* r_2^*(x^* + \varepsilon) < 1.$$

According to Proposition 1 and Proposition 2, there exists $n_0 \in \mathbb{N}$ such that

$$x_* - \varepsilon \leqslant \tilde{x}(n) \leqslant x^* + \varepsilon, \qquad x_* - \varepsilon \leqslant x(n) \leqslant x^* + \varepsilon,$$
$$y_* - \varepsilon \leqslant \tilde{y}(n) \leqslant y^* + \varepsilon, \qquad y_* - \varepsilon \leqslant y(n) \leqslant y^* + \varepsilon$$

for $n \geqslant n_0$.

Notice that $\theta_1(n) \in [0,1]$ implies that $\tilde{x}(n)\exp(\theta_1(n)u(n))$ lies between $\tilde{x}(n)$ and $x(n)$. Similarly, $\tilde{y}(n)\exp(\theta_2(n)v(n))$ lies between $\tilde{y}(n)$ and $y(n)$. From (12), we get

$$\begin{cases} |u(n+1)| \leqslant \max\left\{\left|1 - \frac{r_1^*}{K_{1*}}(x^* + \varepsilon)\right|, \left|1 - \frac{r_{1*}}{K_1^*}(x_* - \varepsilon)\right|\right\}|u(n)| \\ \qquad\qquad + \mu_2^* r_1^*(y^* + \varepsilon)|v(n)|, \\ |v(n+1)| \leqslant \max\left\{\left|1 - \frac{r_2^*}{K_{2*}}(y^* + \varepsilon)\right|, \left|1 - \frac{r_{2*}}{K_2^*}(y_* - \varepsilon)\right|\right\}|v(n)| \\ \qquad\qquad + \mu_1^* r_2^*(x^* + \varepsilon)|u(n)| \end{cases} \tag{14}$$

for $n \geqslant n_0$. Let $\lambda = \max\{\lambda_1^\varepsilon, \lambda_2^\varepsilon\}$. Then $\lambda < 1$. In view of (14), we get

$$\max\{|u(n+1)|, |v(n+1)|\} \leqslant \lambda \max\{|u(n)|, |v(n)|\}, \quad n \geqslant n_0.$$

This implies

$$\max\{|u(n)|, |v(n)|\} \leqslant \lambda^{n-n_0} \max\{|u(n_0)|, |v(n_0)|\}, \quad n \geqslant n_0.$$

Therefore (13) holds and the proof is complete. □

## 4. Concluding remarks and discussions

In this paper, we studied a discrete Lotka–Volterra competition system. Under some reasonable conditions, we showed that the system is persistent. Moreover, we provide a mechanism which models the oscillatory property of the species. That is, when the coefficients are periodic, the system has a periodic solution which is globally stable under some additional conditions.

As mentioned in the introduction, system (1) is the result of coupling of two scalar equations of the form

$$x(n+1) = x(n)\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right]. \tag{15}$$

Eq. (15) has been studied recently by Zhou and Zou [18]. When there is no coupling, that is, $\mu_1(n) \equiv \mu_2(n) \equiv 0$, our persistence result, Theorem 1, reduces to Theorem 2.1 in [18]. Moreover, note that when there is no coupling,

$$1 - \frac{K_1^*}{K_{1*}}\exp\bigl(r_1^* - 1\bigr) = 1 - \frac{r_1^*}{K_{1*}}x^* \leqslant 1 - \frac{r_{1*}}{K_1^*}x_* < 1,$$
$$1 - \frac{K_2^*}{K_{2*}}\exp\bigl(r_2^* - 1\bigr) = 1 - \frac{r_2^*}{K_{2*}}y^* \leqslant 1 - \frac{r_{2*}}{K_2^*}y_* < 1.$$

Thus, $\lambda_1 < 1$ and $\lambda_2 < 1$ are equivalent to

$$\frac{K_1^*}{K_{1*}}\exp\bigl(r_1^* - 1\bigr) < 2 \quad \text{and} \quad \frac{K_2^*}{K_{2*}}\exp\bigl(r_2^* - 1\bigr) < 2.$$

It is shown in [18] that the periodic solution of (15) is globally stable if $\frac{K^*}{K_*}\exp(r^* - 1) \leqslant 2$. Compared with this result, Theorem 3 here almost reduces to Theorem 2.2 in [18] when the coupling disappears. Therefore, we successfully obtain some criteria for the persistence and existence and stability of a periodic solution for system (1) which almost reduce to those for (15) when there is no coupling.

When all the coefficients in (1) are constants, the periodic solution reduces to the equilibrium in the interior of $\mathbb{R}_+^2$. This case is studied by Lu and Wang [11]. Obviously, our results here not only cover Theorem 3 in [11] but also improve it by providing measurement or estimate for the ‘smallness’ of $r_1$ and $r_2$ to guarantee the global stability of the equilibrium point.

Finally, we remark that system (1) was studied recently by Wang et al. [16]. The approach here is different from that in [16] and hence the results obtained here should be quite different from those (Lemma 5 and Corollary 7) there. In fact, this is true. We only give an example to illustrate the persistence (which is called permanence in [16]). Lemma 5 in [16] states that if

$$r_{2*}\left(\frac{r_1}{K_1}\right)_* - r_1^*(r_2\mu_1)^* > 0 \quad \text{and} \quad r_{1*}\left(\frac{r_2}{K_2}\right)_* - r_2^*(r_1\mu_2)^* > 0 \tag{16}$$

then system (1) is persistent. Consider the case where $K_1^* = K_2^* = 1$, $\mu_1^* = 0.5$, $\mu_2^* = 0.4$, $r_1^* = 2.1$, $r_2^* = 2$, $r_{2*} = 1$, $(\frac{r_1}{K_1})_* = 1.1$ and $(r_2\mu_1)^* = 0.8$. Then

$$1 - \mu_1^* x^* = 0.2847 > 0 \quad \text{and} \quad 1 - \mu_2^* y^* = 0.4563 > 0.$$

Therefore, by Theorem 1, system (1) is persistent. However,

$$r_{2*}\left(\frac{r_1}{K_1}\right)_* - r_1^*(r_2\mu_1)^* = -0.58 < 0.$$

That is, condition (16) does not hold. Hence, Lemma 5 [16] is not applicable to system (1) in this case. The example suggests that our results not only complement those in [16] but also may improve them.

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